

TIME-FREQUENCY ANALYSIS OF BORN-JORDAN PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Born-Jordan operators are a class of pseudodifferential operators arising as a generalization of the quantization rule for polynomials on the phase space introduced by Born and Jordan in 1925. The weak definition of such operators involves the Born-Jordan distribution, first introduced by Cohen in 1966 as a member of the Cohen class. We perform a time-frequency analysis of the Cohen kernel of the Born-Jordan distribution, using modulation and Wiener amalgam spaces. We then provide sufficient and necessary conditions for Born-Jordan operators to be bounded on modulation spaces. We use modulation spaces as appropriate symbols classes.

1. INTRODUCTION

In 1925 Born and Jordan [2] introduced for the first time a rigorous mathematical explanation of the notion of “quantization”. This rule was initially restricted to polynomials as symbol classes but was later extended to the class of tempered distribution $\mathcal{S}'(\mathbb{R}^{2d})$ [1, 6]. Roughly speaking, a quantization is a rule which assigns an operator to a function (called symbol) on the phase space \mathbb{R}^{2d} . The Born-Jordan quantization was soon superseded by the most famous Weyl quantization rule proposed by Weyl in [38], giving rise to the well-known Weyl operators (transforms) (see, e.g. [39]).

Recently there has been a regain in interest in the Born-Jordan quantization, both in Quantum Physics and Time-frequency Analysis [17]. The second of us has proved that it is the correct rule if one wants matrix and wave mechanics to be equivalent quantum theories [16]. Moreover, as a time-frequency representation, the Born-Jordan distribution has been proved to be better than the Wigner distribution since it damps very well the unwanted “ghost frequencies”, as shown in [1, 37]. For a throughout and rigorous mathematical explanation of these phenomena we refer to [9] whereas [25, Chapter 5] contains the relevant engineering literature about the geometry of interferences and kernel design.

2010 *Mathematics Subject Classification.* 47G30, 42B35.

Key words and phrases. Time-frequency analysis, Wigner distribution, Born-Jordan distribution, modulation spaces, Wiener amalgam spaces.

To be more specific, the (cross-)Wigner distribution of signals f, g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$(1) \quad W(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \omega} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)} dy.$$

The Weyl operator $\text{Op}_W(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined in terms of the Wigner distribution by the formula

$$\langle \text{Op}_W(a)f, g \rangle = \langle a, W(g, f) \rangle.$$

For $z = (x, \omega)$, consider the Cohen kernel

$$(2) \quad \Theta(z) := \text{sinc}(x\omega) = \begin{cases} \frac{\sin(\pi x\omega)}{\pi x\omega} & \text{for } \omega x \neq 0 \\ 1 & \text{for } \omega x = 0. \end{cases}$$

The (cross-)Born-Jordan distribution $Q(f, g)$ is then defined by

$$(3) \quad Q(f, g) = W(f, g) * \Theta_\sigma, \quad f, g \in \mathcal{S}(\mathbb{R}^d),$$

where Θ_σ is the symplectic Fourier transform of Θ recalled in (21) below. Likewise the Weyl operator, a Born-Jordan operator with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined as

$$(4) \quad \langle \text{Op}_{\text{BJ}}(a)f, g \rangle = \langle a, Q(g, f) \rangle \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Any pseudodifferential operator admits a representation in the Born-Jordan form $\text{Op}_{\text{BJ}}(a)$, as stated in [8].

Now, a first relevant feature of this work is to have computed the Cohen kernel Θ_σ explicitly (cf. the subsequent Proposition 3.4). Namely

$$\Theta_\sigma(\zeta_1, \zeta_2) = \begin{cases} -2 \text{Ci}(4\pi|\zeta_1\zeta_2|), & (\zeta_1, \zeta_2) \in \mathbb{R}^2, d = 1 \\ \mathcal{F}(\chi_{\{|s| \geq 2\}}|s|^{d-2})(\zeta_1, \zeta_2), & (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}, d \geq 2, \end{cases}$$

where $\chi_{\{|s| \geq 2\}}$ is the characteristic function of the set $\{s \in \mathbb{R} : |s| \geq 2\}$ and where

$$\text{Ci}(t) = - \int_t^{+\infty} \frac{\cos s}{s} ds, \quad t \in \mathbb{R}.$$

is the cosine integral function.

This expression of Θ_σ shows that this kernel behaves badly in general: it does not even belong to L_{loc}^∞ (see Corollary 3.5) and has no decay at infinity (see Corollary 3.6). In spite of these facts, it was proved in [9] that some directional smoothing effect is still present, but the analysis carried on there also shows the needed of a systematic and general study of the boundedness of such operators $\text{Op}_{\text{BJ}}(a)$ on modulation spaces, in dependence of the Born-Jordan symbol space. Modulation spaces, introduced by Feichtinger in [19], have been widely employed in the literature to investigate properties of pseudodifferential operators, in particular we

highlight the contributions [3, 4, 14, 24, 28, 31, 32, 33, 34, 35, 36]. For their definition and main properties we refer to the successive section.

The main result concerning the sufficient boundedness conditions of Born-Jordan operators on modulation spaces shows that they behave similarly to Weyl pseudodifferential operators or any other τ -form of pseudodifferential operators. For comparison, see [12, Theorem 5.2, Proposition 5.3], [13, Theorem 1.1] and [35, Theorem 4.3]. The necessary boundedness conditions still contain some open problems, as shown in the following result. We denote q' the conjugate exponent of $q \in [1, \infty]$; it is defined by $1/q + 1/q' = 1$.

Theorem 1.1. *Consider $1 \leq p, q, r_1, r_2 \leq \infty$, such that*

$$(5) \quad p \leq q'$$

and

$$(6) \quad q \leq \min\{r_1, r_2, r'_1, r'_2\}.$$

Then the Born-Jordan operator $\text{Op}_{\text{BJ}}(a)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, having symbol $a \in M^{p,q}(\mathbb{R}^{2d})$, extends uniquely to a bounded operator on $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$, with the estimate

$$(7) \quad \|\text{Op}_{\text{BJ}}(a)f\|_{\mathcal{M}^{r_1, r_2}} \lesssim \|a\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1, r_2}} \quad f \in \mathcal{M}^{r_1, r_2}.$$

Vice-versa, if this conclusion holds true, the constraints (5) is satisfied and it must hold

$$(8) \quad \max \left\{ \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r'_1}, \frac{1}{r'_2} \right\} \leq \frac{1}{q} + \frac{1}{p}$$

which is (6) when $p = \infty$.

Notice that the condition (8) is weaker than (6) when $p < \infty$. The condition (8) is obtained by working with rescaled Gaussians which provide the best localization in terms of Wigner distribution (cf. [29]). On the Fourier side, the Born-Jordan distribution is the point-wise multiplication of the Wigner distribution with the kernel Θ . This reasoning conduces to conjecture that the condition (8) should be the optimal one so that the sufficient boundedness conditions for Born-Jordan operators might be weaker than the corresponding ones for Weyl and τ -pseudodifferential operators. But the matter is really subtle and requires a new and most refined analysis of the kernel Θ . In particular the zeroes of the Θ function should play a key for a thorough understanding of such operators, which certainly deserve further study.

The paper is organized as follows. Section 2 is devoted to some preliminary results from Time-frequency Analysis. In Section 3 we perform an analysis of the kernel Θ and we prove the above formula for Θ_σ . In Sections 4 and 5 we study the

Cohen kernels and the boundedness of Born-Jordan operators in the framework of modulation spaces.

2. PRELIMINARIES

In this section we recall the definition of the spaces involved in our study and present the main time-frequency tools used.

Modulation and Wiener amalgam spaces. The modulation and Wiener amalgam space norms are a measure of the joint time-frequency distribution of $f \in \mathcal{S}'$. For their basic properties we refer to the original literature [18, 19, 20] and the textbooks [15, 23].

Let $f \in \mathcal{S}'(\mathbb{R}^d)$. We define the short-time Fourier transform of f as

$$(9) \quad V_g f(z) = \langle f, \pi(z)g \rangle = \mathcal{F}[fT_x g](\omega) = \int_{\mathbb{R}^d} f(y) \overline{g(y-x)} e^{-2\pi i y \omega} dy$$

for $z = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$.

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M^{p,q}$ is

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p (x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/p}$$

(with natural modifications when $p = \infty$ or $q = \infty$). If $p = q$, we write M^p instead of $M^{p,p}$.

The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g , in the sense that different nonzero window functions yield equivalent norms. The modulation space $M^{\infty,1}$ is also called Sjöstrand's class [31].

The closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M^{p,q}$ -norm is denoted $\mathcal{M}^{p,q}(\mathbb{R}^d)$. Then

$$\mathcal{M}^{p,q}(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d), \quad \text{and } \mathcal{M}^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d),$$

provided $p < \infty$ and $q < \infty$.

Recalling that the conjugate exponent p' of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$, for any $p, q \in [1, \infty]$ the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear map $M^{p,q}(\mathbb{R}^d) \times M^{p',q'}(\mathbb{R}^d) \rightarrow \mathbb{C}$.

Modulation spaces enjoy the following inclusion properties:

$$(10) \quad \mathcal{S}(\mathbb{R}^d) \subseteq M^{p_1, q_1}(\mathbb{R}^d) \subseteq M^{p_2, q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d), \quad p_1 \leq p_2, \quad q_1 \leq q_2.$$

The Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ are given by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p d\omega \right)^{q/p} dx \right)^{1/q} < \infty$$

(with obvious changes for $p = \infty$ or $q = \infty$). Using Parseval identity in (9), we can write the so-called fundamental identity of time-frequency analysis $V_g f(x, \omega) = e^{-2\pi i x \omega} V_{\hat{g}} \hat{f}(\omega, -x)$, hence

$$|V_g f(x, \omega)| = |V_{\hat{g}} \hat{f}(\omega, -x)| = |\mathcal{F}(\hat{f} T_{\omega} \bar{\hat{g}})(-x)|$$

so that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f} T_{\omega} \bar{\hat{g}}\|_{\mathcal{F}L^p}^q(\omega) d\omega \right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L^p, L^q)}.$$

This means that these Wiener amalgam spaces are simply the image under Fourier transform of modulation spaces:

$$(11) \quad \mathcal{F}(M^{p,q}) = W(\mathcal{F}L^p, L^q).$$

We will often use the following product property of Wiener amalgam spaces ([18, Theorem 1 (v)], Theorem 1 (v)):

$$(12) \quad f \in W(\mathcal{F}L^1, L^\infty) \text{ and } g \in W(\mathcal{F}L^p, L^q) \implies fg \in W(\mathcal{F}L^p, L^q).$$

In order to prove the necessary boundedness conditions for Born-Jordan operators we shall use the dilation properties for Gaussian functions. Precisely, consider $\varphi(x) = e^{-\pi|x|^2}$ and define

$$(13) \quad \varphi_\lambda(x) = \varphi(\sqrt{\lambda}x) = e^{-\pi\lambda|x|^2}, \quad \lambda > 0.$$

The dilation properties for the Gaussian φ_λ in modulation spaces were proved in [35, Lemma 1.8] (see also [7, Lemma 3.2]).

Lemma 2.1. *For $1 \leq p, q \leq \infty$, we have*

$$(14) \quad \|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-\frac{d}{2q'}} \quad \text{as } \lambda \rightarrow +\infty$$

$$(15) \quad \|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-\frac{d}{2p}} \quad \text{as } \lambda \rightarrow 0^+.$$

The following dilation properties are a straightforward generalization of [9, Lemma 2.3].

Lemma 2.2. *Consider $1 \leq p, q \leq \infty$, $\psi \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ and $\lambda > 0$. Then*

$$(16) \quad \|\psi(\sqrt{\lambda} \cdot)\|_{W(\mathcal{F}L^p, L^q)} \asymp \lambda^{-\frac{d}{2p'}} \quad \text{as } \lambda \rightarrow +\infty$$

$$(17) \quad \|\psi(\sqrt{\lambda} \cdot)\|_{W(\mathcal{F}L^p, L^q)} \asymp \lambda^{-\frac{d}{2q}} \quad \text{as } \lambda \rightarrow 0^+.$$

The same conclusion holds uniformly with respect to λ if ψ varies in bounded subsets of $C_c^\infty(\mathbb{R}^d)$.

Another tool for obtaining the optimality of our results is the cross-Wigner distribution of rescaled Gaussian functions. The proof is a straightforward computation (see Prop. 244 in [15]):

Lemma 2.3. *Consider $\varphi(x) = e^{-\pi|x|^2}$ and φ_λ as in (13). Then*

$$(18) \quad W(\varphi, \varphi_\lambda)(x, \omega) = \frac{2^d}{(\lambda + 1)^{\frac{d}{2}}} e^{-\frac{4\pi\lambda}{\lambda+1}|x|^2} e^{-\frac{4\pi}{\lambda+1}|\omega|^2} e^{-4\pi i \frac{\lambda-1}{\lambda+1}x\omega}.$$

It follows that:

Corollary 2.4. *Consider φ and φ_λ as in the assumptions of Lemma 2.3. Then*

$$(19) \quad \mathcal{FW}(\varphi, \varphi_\lambda)(\zeta_1, \zeta_2) = \frac{1}{(\lambda + 1)^{\frac{d}{2}}} e^{-\frac{\pi}{\lambda+1}\zeta_1^2} e^{-\frac{\pi\lambda}{\lambda+1}\zeta_2^2} e^{-\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}.$$

Proof. Formula (19) is easily obtained from (18) using well-known Gaussian integral formulas; it can also be painlessly obtained from (18) by observing that for any functions $\psi, \phi \in L^2(\mathbb{R}^d)$ the following relation between the cross-Wigner distribution and its Fourier transform holds:

$$\mathcal{FW}(\psi, \phi)(x, \omega) = 2^{-d} W(\psi, \phi^\vee)(\tfrac{1}{2}\omega, -\tfrac{1}{2}x)$$

where $\phi^\vee(x) = \phi(-x)$ (see formula (9.27) in [15], or formula (1.90) in Folland [22]). \square

We denote by σ the symplectic form on the phase space $\mathbb{R}^{2d} \equiv \mathbb{R}^d \times \mathbb{R}^d$; the phase space variable is denoted $z = (x, \omega)$ and the dual variable by $\zeta = (\zeta_1, \zeta_2)$. By definition $\sigma(z, \zeta) = Jz \cdot \zeta = \omega \cdot \zeta_1 - x \cdot \zeta_2$, where

$$(20) \quad J = \begin{pmatrix} 0_{d \times d} & I_{d \times d} \\ -I_{d \times d} & 0_{d \times d} \end{pmatrix}.$$

The Fourier transform of a function f on \mathbb{R}^d is normalized as

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x \omega} f(x) dx,$$

and the symplectic Fourier transform of a function F on the phase space \mathbb{R}^{2d} is

$$(21) \quad \mathcal{F}_\sigma F(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta, z)} F(z) dz.$$

Observe that $\mathcal{F}_\sigma F(\zeta) = \mathcal{F}F(J\zeta)$. Hence the symplectic Fourier transform of the Wigner distribution (18) is given by

$$(22) \quad \mathcal{F}_\sigma W(\varphi, \varphi_\lambda)(\zeta_1, \zeta_2) = \frac{1}{(\lambda + 1)^{\frac{d}{2}}} e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2} e^{-\frac{\pi}{\lambda+1}\zeta_2^2} e^{\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}.$$

We will also use the important relation

$$(23) \quad \mathcal{F}_\sigma[F * G] = \mathcal{F}_\sigma F \mathcal{F}_\sigma G.$$

The convolution relations for modulation spaces are essential in the proof of the boundedness of a Born-Jordan operator on these spaces and were proved in [10, Proposition 2.4]:

Proposition 2.1. *Let $1 \leq p, q, r, s, t \leq \infty$. If*

$$\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \quad \frac{1}{t} + \frac{1}{t'} = 1,$$

then

$$(24) \quad M^{p,st}(\mathbb{R}^d) * M^{q,st'}(\mathbb{R}^d) \hookrightarrow M^{r,s}(\mathbb{R}^d)$$

*with $\|f * h\|_{M^{r,s}} \lesssim \|f\|_{M^{p,st}} \|h\|_{M^{q,st'}}$.*

We also recall the useful result proved in [9, Lemma 5.1].

Lemma 2.5. *Let $\chi \in C_c^\infty(\mathbb{R})$. Then, for $\zeta_1, \zeta_2 \in \mathbb{R}^d$, the function $\chi(\zeta_1 \zeta_2)$ belongs to $W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$.*

3. ANALYSIS OF THE COHEN KERNEL Θ

Consider the Cohen kernel Θ defined in (2). Obviously $\Theta \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ but displays a vary bad decay at infinity, as clarified in what follows.

Proposition 3.1. *For $1 \leq p < \infty$, the function $\Theta \notin L^p(\mathbb{R}^{2d})$.*

Proof. Observe that, for $t \in \mathbb{R}$, $|t| \leq 1/2$, the function $\text{sinc } t$ satisfies $|\text{sinc } t| \geq 2/\pi$. Hence, for any $1 \leq p < \infty$,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |\Theta(x, \omega)|^p dx d\omega &= \int_{\mathbb{R}^{2d}} |\text{sinc}(x\omega)|^p dx d\omega \\ &\geq \int_{|x\omega| \leq 1/2} |\text{sinc}(x\omega)|^p dx d\omega \\ &\geq \left(\frac{2}{\pi}\right)^p \int_{|x\omega| \leq 1/2} dx d\omega \\ &= \left(\frac{2}{\pi}\right)^p \text{meas}\{(x, \omega) : |x\omega| \leq 1/2\} = +\infty. \end{aligned}$$

This concludes the proof. \square

We continue our investigation of the function Θ by looking for the right Wiener amalgam and modulation spaces containing this function. For this reason, we first reckon explicitly the STFT of the Θ function, with respect to the Gaussian window $g(x, \omega) = e^{-\pi x^2} e^{-\pi \omega^2} \in \mathcal{S}(\mathbb{R}^{2d})$.

Proposition 3.2. *For $z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{R}^d$,*

(25)

$$V_g \Theta(z_1, z_2, \zeta_1, \zeta_2)$$

(26)

$$= \int_{-1/2}^{1/2} \frac{1}{(t^2 + 1)^{d/2}} e^{-2\pi i [\frac{1}{t} \zeta_1 \zeta_2 + \frac{t}{t^2+1} (z_1 - \frac{1}{t} \zeta_2)(z_2 - \frac{1}{t} \zeta_1)]} e^{-\pi \frac{t^2}{t^2+1} [(z_1 - \frac{1}{t} \zeta_2)^2 + (z_2 - \frac{1}{t} \zeta_1)^2]} dt.$$

Proof. We write $\Theta(z_1, z_2) = F_1(z_1, z_2) + F_2(z_1, z_2)$, where $F_1(z_1, z_2) = \int_0^{1/2} e^{2\pi i z_1 z_2 t} dt$ and $F_2(z) = F_1(Jz)$, $z = (z_1, z_2)$. Let us first reckon $V_g F_1(z, \zeta)$, $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, where g is the Gaussian function above. For $t > 0$ we define the function $H_t(z_1, z_2) = e^{2\pi i t z_1 z_2}$ and observe that

$$(27) \quad \mathcal{F} H_t(\zeta_1, \zeta_2) = \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2}$$

(cf. [22, Appendix A, Theorem 2]). By the Dominated Convergence Theorem,

$$\begin{aligned}
V_g F_1(z, \zeta) &= \int_0^{1/2} \mathcal{F}(H_t T_z g)(\zeta) dt = \int_0^{1/2} (\mathcal{F}(H_t) * M_{-z} \hat{g})(\zeta_1, \zeta_2) dt \\
&= \int_0^{1/2} \frac{1}{t^d} \int_{\mathbb{R}^{2d}} e^{-2\pi i \frac{1}{t} (\zeta_1 - y_1) \cdot (\zeta_2 - y_2)} e^{-2\pi i (z_1, z_2) \cdot (y_1, y_2)} e^{-\pi y_1^2} e^{-\pi y_2^2} dy_1 dy_2 dt \\
&= \int_0^{1/2} \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} \int_{\mathbb{R}^{2d}} e^{-2\pi i \frac{1}{t} y_1 y_2 + 2\pi i \frac{1}{t} (\zeta_2 y_1 + \zeta_1 y_2) - 2\pi i (z_1 y_1 + z_2 y_2)} e^{-\pi (y_1^2 + y_2^2)} dy_1 dy_2 dt \\
&= \int_0^{1/2} \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} \int_{\mathbb{R}^d} e^{2\pi i (\frac{1}{t} \zeta_1 y_2 - z_2 y_2)} e^{-\pi y_2^2} \\
&\quad \cdot \left(\int_{\mathbb{R}^d} e^{-2\pi i y_1 \cdot (\frac{1}{t} y_2 - \frac{1}{t} \zeta_2 + z_1)} e^{-\pi y_1^2} dy_1 \right) dy_2 dt \\
&= \int_0^{1/2} \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} \int_{\mathbb{R}^d} e^{-2\pi i y_2 \cdot (z_2 - \frac{1}{t} \zeta_1)} e^{-\pi y_2^2} e^{-\pi (\frac{1}{t} y_2 - \frac{1}{t} \zeta_2 + z_1)^2} dy_2 dt \\
&= \int_0^{1/2} \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} e^{-\pi (z_1 - \frac{1}{t} \zeta_2)^2} \int_{\mathbb{R}^d} e^{-2\pi i y_2 \cdot (z_2 - \frac{1}{t} \zeta_1)} e^{-\pi ((1 + \frac{1}{t^2}) y_2^2 - 2(z_1 - \frac{1}{t} \zeta_2) \cdot \frac{1}{t} y_2)} dy_2 dt \\
&= \int_0^{1/2} \frac{1}{t^d} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} e^{-\pi \frac{t^2}{t^2+1} (z_1 - \frac{1}{t} \zeta_2)^2} \int_{\mathbb{R}^d} e^{-2\pi i y_2 \cdot (z_2 - \frac{1}{t} \zeta_1)} \\
&\quad \cdot e^{-\pi (\frac{\sqrt{t^2+1}}{t} y_2 - \frac{t}{\sqrt{t^2+1}} (z_1 - \frac{1}{t} \zeta_2))^2} dy_2 dt \\
&= \int_0^{1/2} \frac{1}{(t^2 + 1)^{d/2}} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} e^{-\pi \frac{t^2}{t^2+1} (z_1 - \frac{1}{t} \zeta_2)^2} \\
&\quad \cdot \int_{\mathbb{R}^d} e^{-2\pi i (\frac{t}{\sqrt{t^2+1}} w + \frac{t}{t^2+1} (z_1 - \frac{1}{t} \zeta_2)) (z_2 - \frac{1}{t} \zeta_1)} e^{-\pi w^2} dw dt \\
&= \int_0^{1/2} \frac{1}{(t^2 + 1)^{d/2}} e^{-2\pi i \frac{1}{t} \zeta_1 \zeta_2} e^{-\pi \frac{t^2}{t^2+1} (z_1 - \frac{1}{t} \zeta_2)^2} e^{-\pi \frac{t^2}{t^2+1} (z_2 - \frac{1}{t} \zeta_1)^2} \\
&\quad \cdot e^{-2\pi i \frac{t}{t^2+1} (z_1 - \frac{1}{t} \zeta_2) (z_2 - \frac{1}{t} \zeta_1)} dt.
\end{aligned}$$

Now, an easy computation shows

$$V_g F_2(z, \zeta) = V_g F_1(Jz, J\zeta)$$

so that $V_g \Theta = V_g F_1 + V_g F_2$ and we obtain (26). \square

Proposition 3.3. *The function Θ in (2) belongs to $W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$.*

Proof. We simply have to calculate

$$\sup_{z \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |V_g \Theta(z, \zeta)| d\zeta.$$

From (26) we observe that

$$\begin{aligned}
\|V_g\Theta(z, \cdot)\|_1 &\leq \int_{-1/2}^{1/2} \int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi \frac{t^2}{t^2+1}(z_1 - \frac{1}{t}\zeta_2)^2} e^{-\pi \frac{t^2}{t^2+1}(z_2 - \frac{1}{t}\zeta_1)^2} d\zeta_1 d\zeta_2 dt \\
&= \int_{-1/2}^{1/2} \int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi \frac{1}{t^2+1}(tz_1 - \zeta_2)^2} e^{-\pi \frac{1}{t^2+1}(tz_2 - \zeta_1)^2} d\zeta_1 d\zeta_2 dt \\
&= \int_{-1/2}^{1/2} \int_{\mathbb{R}^{2d}} (t^2+1)^{d/2} e^{-\pi(v_1^2+v_2^2)} dv_1 dv_2 dt = C < \infty,
\end{aligned}$$

from which the claim follows. \square

Using the STFT of the function Θ in (26) we observe that

$$\|V_g\Theta(\cdot, \zeta)\|_1 \leq \int_{-1/2}^{1/2} \int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi \frac{t^2}{t^2+1}(u_1 - \frac{1}{t}\zeta_2)^2} e^{-\pi \frac{t^2}{t^2+1}(u_2 - \frac{1}{t}\zeta_1)^2} du_1 du_2 dt = +\infty$$

so that we conjecture that $\Theta \notin M^{1,\infty}(\mathbb{R}^{2d})$. The previous claim will follow if we prove that $\Theta_\sigma \notin W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$.

Note that $\Theta_\sigma(\zeta) = \mathcal{F}\Theta(J\zeta) = \mathcal{F}\Theta(\zeta)$. Furthermore, the distributional Fourier transform of Θ can be computed explicitly as follows. First, recall the definition of the cosine integral function (1).

Proposition 3.4. *For $d \geq 1$ the distribution symplectic Fourier transform Θ_σ of the function Θ is provided by*

$$(28) \quad \Theta_\sigma(\zeta_1, \zeta_2) = \begin{cases} -2 \operatorname{Ci}(4\pi|\zeta_1\zeta_2|), & (\zeta_1, \zeta_2) \in \mathbb{R}^2, d = 1 \\ \mathcal{F}(\chi_{\{|s| \geq 2\}}|s|^{d-2})(\zeta_1, \zeta_2), & (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}, d \geq 2, \end{cases}$$

where $\chi_{\{|s| \geq 2\}}$ is the characteristic function of the set $\{s \in \mathbb{R} : |s| \geq 2\}$. The case $d = 1$ can be recaptured by the case $d \geq 2$ using (1).

Proof. We carry out the computations of Θ_σ by studying first the case in dimension $d = 1$ and secondly, inspired by the former case, $d > 1$.

First step: $d = 1$. By Proposition 3.1, the function Θ is in

$$L^\infty(\mathbb{R}^2) \setminus L^p(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2), \quad 1 \leq p < \infty,$$

so that the Fourier transform is meant in $\mathcal{S}'(\mathbb{R}^2)$. Observe that

$$\mathcal{F}\Theta(\zeta_1, \zeta_2) = \mathcal{F}_2\mathcal{F}_1\Theta(\zeta_1, \zeta_2),$$

where \mathcal{F}_1 (resp. \mathcal{F}_2) is the partial Fourier transform with respect to the first (resp. second) variable. Indeed, for every test function $\varphi \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathcal{F}\Theta, \varphi \rangle = \langle \Theta, \mathcal{F}^{-1}\varphi \rangle$$

and $\mathcal{F}^{-1}\varphi(x, \omega) = \mathcal{F}_1^{-1}\mathcal{F}_2^{-1}\varphi(x, \omega) = \mathcal{F}_2^{-1}\mathcal{F}_1^{-1}\varphi(x, \omega)$, by Fubini's Theorem.

Using

$$\mathcal{F}_1 \text{sinc}(y_2 \cdot)(\zeta_1) = \frac{1}{|y_2|} p_{1/2}(\zeta_1/y_2), \quad y_2 \neq 0,$$

where $p_{1/2}(t)$ is the box function defined by $p_{1/2}(t) = 1$ for $|t| \leq 1/2$ and $p_{1/2}(t) = 0$ otherwise, we obtain, for $\zeta_1 \zeta_2 \neq 0$ (hence in particular $|\zeta_1| > 0$),

$$\begin{aligned} \mathcal{F}\Theta(\zeta_1, \zeta_2) &= \int_{\mathbb{R}} e^{-2\pi i \zeta_2 y_2} \frac{1}{|y_2|} p_{1/2}(\zeta_1/y_2) dy_2 = \int_{|y_2| \geq 2|\zeta_1|} e^{-2\pi i \zeta_2 y_2} \frac{1}{|y_2|} dy_2 \\ &= \int_{|s| \geq 2|\zeta_1 \zeta_2|} e^{-2\pi i s} \frac{1}{|s|} ds \\ &= \int_{|s| \geq 2|\zeta_1 \zeta_2|} \frac{\cos(2\pi s) - i \sin(2\pi s)}{|s|} ds \\ &= \int_{|s| \geq 2|\zeta_1 \zeta_2|} \frac{\cos 2\pi s}{|s|} ds \\ &= 2 \int_{2|\zeta_1 \zeta_2|}^{+\infty} \frac{\cos 2\pi s}{s} ds = -2\text{Ci}(4\pi|\zeta_1 \zeta_2|), \end{aligned}$$

by (1), so that, since $\zeta_1 \zeta_2 = 0$ is a set of Lebesgue measure equal zero on \mathbb{R}^2 , we can write

$$(29) \quad \Theta_\sigma(\zeta_1, \zeta_2) = -2\text{Ci}(4\pi|\zeta_1 \zeta_2|), \quad (\zeta_1, \zeta_2) \in \mathbb{R}^2.$$

Second step: $d > 1$. This is a simple generalization on the former step. For $(z_1, z_2), (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, $d > 1$, we write

$$(30) \quad z_i = (z'_i, z_{i,d}), \quad \zeta_i = (\zeta'_i, \zeta_{i,d}), \quad z'_i, \zeta'_i \in \mathbb{R}^{d-1}, \quad z_{i,d}, \zeta_{i,d} \in \mathbb{R}, \quad i = 1, 2.$$

We decompose $\mathcal{F}\Theta = \mathcal{F}_{2d} \mathcal{F}' \mathcal{F}_1 \Theta$ where, for $\Theta = \Theta(z_1, z_2)$, \mathcal{F}_1 is the partial Fourier transform with respect to the variable $z_{1,d}$, \mathcal{F}' is the partial Fourier transform with respect the $2d-2$ variables $(z'_1, z'_2) \in \mathbb{R}^{2d-2}$ and \mathcal{F}_{2d} is the partial Fourier transform with respect to the last variable $z_{2,d}$. We start with computing the partial Fourier transform \mathcal{F}_1 :

$$\begin{aligned} \mathcal{F}_1 \Theta(z'_1, \cdot, z'_2, z_{2,d})(\zeta_{1,d}) &= \mathcal{F}_1 (T_{\frac{-z'_1 z'_2}{z_{2,d}}} \text{sinc}(z_{2,d} \cdot))(\zeta_{1,d}) \\ &= e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z'_1 z'_2} \frac{1}{|z_{2,d}|} \mathcal{F}_1(\text{sinc}) \left(\frac{\zeta_{1,d}}{z_{2,d}} \right) \\ &= e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z'_1 z'_2} \frac{1}{|z_{2,d}|} p_{1/2} \left(\frac{\zeta_{1,d}}{z_{2,d}} \right). \end{aligned}$$

Using the Gaussian integrals in [22, Appendix A, Theorem 2]) we calculate

$$\mathcal{F}'(e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z'_1 z'_2})(\zeta'_1, \zeta'_2) = \left| \frac{z_{2,d}}{\zeta_{1,d}} \right|^{d-1} e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta'_1 \zeta'_2},$$

so that

$$\begin{aligned} \mathcal{F}\Theta(\zeta_1, \zeta_2) &= \mathcal{F}_{2d} \left(e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta'_1 \zeta'_2} \left| \frac{z_{2,d}}{\zeta_{1,d}} \right|^{d-1} \frac{1}{|z_{2,d}|} p_{1/2} \left(\frac{\zeta_{1,d}}{z_{2,d}} \right) \right) (\zeta_{2,d}) \\ &= \int_{\left| \frac{\zeta_{1,d}}{z_{2,d}} \right| \leq \frac{1}{2}} \left| \frac{z_{2,d}}{\zeta_{1,d}} \right|^{d-1} \frac{1}{|z_{2,d}|} e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta_1 \zeta_2} dz_{2,d} \\ &= \int_{|s| \geq 2} e^{-2\pi i s(\zeta_1 \zeta_2)} |s|^{d-2} ds, \end{aligned}$$

as claimed. \square

Notice that the second equation (28) can be written

$$\Theta_\sigma(\zeta_1, \zeta_2) = \int_{|s| \geq 2} e^{-2\pi i s(\zeta_1 \zeta_2)} |s|^{d-2} ds.$$

Corollary 3.5. *We have*

$$\Theta_\sigma \notin L_{loc}^\infty(\mathbb{R}^{2d}).$$

Proof. For the case $d = 1$, recall that the cosine integral $\text{Ci}(x)$ has the series expansion

$$\text{Ci}(x) = \gamma + \log x + \sum_{k=1}^{+\infty} \frac{(-x^2)^k}{2k(2k)!}, \quad x > 0$$

where γ is the Euler–Mascheroni constant, from which our claim easily follows.

For $d \geq 2$, Θ_σ is only defined as a tempered distribution. \square

Corollary 3.6. *The function $\Theta_\sigma \notin L^p(\mathbb{R}^{2d})$, for any $1 \leq p \leq \infty$.*

Proof. The case $p = \infty$ is already treated in Corollary 3.5. For $d \geq 2$ again we observe that Θ_σ is not defined as function but only as a tempered distribution. For $d = 1$, $1 \leq p < \infty$, the claim follows by the expression (29). Indeed, choose $0 < \epsilon < \pi/2$, then $|\text{Ci}(x)| \geq |\text{Ci}(\epsilon)|$, for $0 < x < \epsilon$, so that

$$\begin{aligned} \int_{\mathbb{R}^2} |\Theta_\sigma(\zeta_1, \zeta_2)|^p d\zeta_1 d\zeta_2 &\geq 2 \int_{|\zeta_1 \zeta_2| < \frac{\epsilon}{4\pi}} |\text{Ci}(4\pi|\zeta_1 \zeta_2|)|^p d\zeta d\zeta_2 \\ &\geq C_p \text{meas}\{(\zeta_1, \zeta_2) : |\zeta_1 \zeta_2| < \frac{\epsilon}{4\pi}\} = +\infty, \end{aligned}$$

for a suitable constant $C_p > 0$. \square

Since $\mathcal{FL}^1 \subset L^\infty$, the Wiener amalgam space $W(\mathcal{FL}^1, L^\infty)$ is included in L_{loc}^∞ . This proves our claim:

Corollary 3.7. *The function $\Theta_\sigma \notin W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$ or, equivalently, $\Theta \notin M^{1,\infty}(\mathbb{R}^{2d})$.*

4. COHEN KERNELS IN MODULATION AND WIENER SPACES

In this section we focus on other members of the Cohen class, introduced by Cohen in [5], which define, for $\tau \in [0, 1]$, the (cross-) τ -Wigner distributions

$$(31) \quad W_\tau(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \zeta} f(x + \tau y) \overline{g(x - (1 - \tau)y)} dy \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

Such distributions enter in the definition of the τ -pseudodifferential operators as follows

$$(32) \quad \langle \text{Op}_\tau(a) f, g \rangle = \langle a, W_\tau(g, f) \rangle \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

It is then natural to investigate the time-frequency properties of such kernels and compare to the corresponding Weyl and Born-Jordan ones. The Cohen class consists of elements of the type

$$M(f, f)(x, \omega) = W(f, f) * \sigma$$

where $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is called the Cohen kernel. When $\sigma = \delta$, then $M(f, f) = W(f, f)$ and we come back to the Wigner distribution. When $\sigma = \Theta_\sigma$, then $M(f, f) = Q(f, f)$, that is the Born-Jordan distribution. The τ -Wigner function $W_\tau(f, f)$ belongs to the Cohen class for every $\tau \in [0, 1]$, as proved in [1, Proposition 5.6]:

$$W_\tau(f, f) = W(f, f) * \sigma_\tau,$$

where

$$\sigma_\tau(x, \omega) = \frac{2^d}{|2\tau - 1|^d} e^{2\pi i \frac{2}{2\tau - 1} x \omega}, \quad \tau \neq \frac{1}{2}$$

and $\sigma_{1/2} = \delta$ (the case of the Wigner distribution, as already observed).

In what follows we study the properties of the Cohen kernels σ_τ in the realm of modulation and Wiener amalgam spaces. As we shall see, the Born-Jordan kernel and the Wigner one display similar time-frequency properties and are locally worse than the kernels σ_τ , $\tau \neq 1/2$.

Proposition 4.1. *We have, for every $\tau \in [0, 1] \setminus \{1/2\}$,*

$$\sigma_\tau \in W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d}) \cap M^{1,\infty}(\mathbb{R}^{2d}).$$

Proof. We exploit the dilation properties for Wiener spaces (cf. [33, Lemma 3.2] and its generalization in [7, Corollary 3.2]): for $A = \lambda I$, $\lambda > 0$,

$$(33) \quad \|f(A \cdot)\|_{W(\mathcal{FL}^p, L^q)} \leq C \lambda^{d(\frac{1}{p} - \frac{1}{q} - 1)} (\lambda^2 + 1)^{d/2} \|f\|_{W(\mathcal{FL}^p, L^q)}.$$

Using the dilation relations for Wiener amalgam spaces (33) for $\lambda = \sqrt{t}$, $0 < t < 1/2$, $p = 1$, $q = \infty$, we obtain

$$\|e^{\pm 2\pi i \zeta_1 \zeta_2 t}\|_{W(\mathcal{FL}^1, L^\infty)} \leq C \|e^{\pm 2\pi i \zeta_1 \zeta_2}\|_{W(\mathcal{FL}^1, L^\infty)}$$

with constant $C > 0$ independent on the parameter t . For $t = \frac{2}{2\tau-1}$, when $\tau > 1/2$ and $t = -\frac{2}{2\tau-1}$, when $0 \leq \tau < 1/2$, we obtain that $\sigma_\tau \in W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$. Now, an easy computation gives

$$\mathcal{F}\sigma_\tau(\zeta_1, \zeta_2) = e^{-\pi i(2\tau-1)\zeta_1 \zeta_2},$$

so that, using $\mathcal{FM}^{1,\infty}(\mathbb{R}^{2d}) = W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$ and repeating the previous argument we obtain $\sigma_\tau \in M^{1,\infty}(\mathbb{R}^{2d})$ for every $\tau \in [0, 1] \setminus \{1/2\}$. \square

The case $\tau = 1/2$ is different. Indeed, $\sigma_{1/2} = \delta$ and for any fixed $g \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$ the STFT $V_g \delta$ is given by

$$V_g \delta(z, \zeta) = \langle \delta, M_\zeta T_z g \rangle = \overline{g(-z)},$$

that yields $\delta \in M^{1,\infty}(\mathbb{R}^{2d}) \setminus W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$.

The Born-Jordan kernel Θ_σ behaves analogously. Indeed, using Proposition 3.3 and Corollary 3.7, we obtain

$$\Theta_\sigma \in M^{1,\infty}(\mathbb{R}^{2d}) \setminus W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d}).$$

Those distributions can be used in the definition of the τ -pseudodifferential operators

5. SYMBOLS IN MODULATION SPACES

This section is focused on the proof of Theorem 1.1. We first demonstrate the sufficient boundedness conditions.

Theorem 5.1. *Assume that $1 \leq p, q, r_1, r_2 \leq \infty$. Then the pseudodifferential operator $\text{Op}_{\text{BJ}}(a)$, from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, having symbol $a \in M^{p,q}(\mathbb{R}^{2d})$, extends uniquely to a bounded operator on $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$, with the estimate (7) and the indices' conditions (5) and (6).*

The result relies on a thorough understanding the action of the mapping

$$(34) \quad A : a \longmapsto a * \Theta_\sigma,$$

which gives the Weyl symbol of an operator with Born-Jordan symbol a , on modulation space s .

Proposition 5.1. *For every $1 \leq p, q \leq \infty$, the mapping A in (34), defined initially on $\mathcal{S}'(\mathbb{R}^{2d})$, restricts to a linear continuous map on $M^{p,q}(\mathbb{R}^{2d})$, i.e., there exists a constant $C > 0$ such that*

$$(35) \quad \|Aa\|_{M^{p,q}} \leq C \|a\|_{M^{p,q}}.$$

Proof. By Proposition (3.3), the function $\Theta \in W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d})$. Its symplectic Fourier transform Θ_σ belongs to $\mathcal{F}_\sigma W(\mathcal{FL}^1, L^\infty)(\mathbb{R}^{2d}) = M^{1,\infty}(\mathbb{R}^{2d})$. Now, for every $1 \leq p, q \leq \infty$, the convolution relations for modulation space \mathcal{S} (24) give

$$M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d})$$

and this shows the claim (35). \square

Proof of Theorem 5.1. Assume $a \in M^{p,q}(\mathbb{R}^{2d})$, then Proposition 5.1 proves that $Aa = a * \Theta_\sigma \in M^{p,q}(\mathbb{R}^{2d})$ as well. We next write $\text{Op}_{\text{BJ}}(a) = \text{Op}_W(Aa)$ and use the sufficient conditions for Weyl operators in [12, Theorem 5.2]: if the Weyl symbol Aa is in $M^{p,q}(\mathbb{R}^{2d})$, then $\text{Op}_W(Aa)$ extends to a bounded operator on $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$, with

$$\|\text{Op}_{\text{BJ}}(a)f\|_{\mathcal{M}^{r_1, r_2}} = \|\text{Op}_W(Aa)f\|_{\mathcal{M}^{r_1, r_2}} \lesssim \|Aa\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1, r_2}}$$

where the indices r_1, r_2, p, q satisfy (5) and (6). The inequality (35) then provides the claim. \square

The necessary conditions of Theorem 1.1 require some preliminaries.

We reckon the adjoint operator $\text{Op}_{\text{BJ}}(a)^*$ of a Born-Jordan operator $\text{Op}_{\text{BJ}}(a)$ using the connection with Weyl operators. Recall that $\text{Op}_W(b)^* = \text{Op}_W(\bar{b})$ [26], so that

$$\text{Op}_{\text{BJ}}(a)^* = \text{Op}_W(a * \Theta_\sigma)^* = \text{Op}_W(\overline{a * \Theta_\sigma}) = \text{Op}_W(\bar{a} * \bar{\Theta}_\sigma) = \text{Op}_W(\bar{a} * \Theta_\sigma) = \text{Op}_{\text{BJ}}(\bar{a})$$

because Θ is an even real-valued function. Hence the adjoint of a Born-Jordan operator $\text{Op}_{\text{BJ}}(a)$ with symbol a is the Born-Jordan operator having symbol \bar{a} (the complex-conjugate of a). This nice property is the key argument for the following auxiliary result, already obtained for the case of Weyl operators in [12, Lemma 5.1]. The proof uses the same pattern as the former result and hence is omitted.

Lemma 5.2. *Suppose that, for some $1 \leq p, q, r_1, r_2 \leq \infty$, the following estimate holds:*

$$\|\text{Op}_{\text{BJ}}(a)f\|_{M^{r_1, r_2}} \leq C \|a\|_{M^{p,q}} \|f\|_{M^{r_1, r_2}}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Then the same estimate is satisfied with r_1, r_2 replaced by r'_1, r'_2 (even if $r_1 = \infty$ or $r_2 = \infty$).

The above instruments let us show the necessity of (5) and (8).

Theorem 5.2. *Suppose that, for some $1 \leq p, q, r_1, r_2 \leq \infty$, $C > 0$ the estimate*

$$(36) \quad \|\text{Op}_{\text{BJ}}(a)f\|_{M^{r_1, r_2}} \leq C \|a\|_{M^{p,q}} \|f\|_{M^{r_1, r_2}} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \quad f \in \mathcal{S}(\mathbb{R}^d)$$

holds. Then the constraints in (5) and (8) must hold.

Proof. The estimate (36) can be written as

$$|\langle a, Q(f, g) \rangle| \leq C \|a\|_{M^{p,q}} \|f\|_{M^{r_1, r_2}} \|g\|_{M^{r'_1, r'_2}} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \quad f, g \in \mathcal{S}(\mathbb{R}^d)$$

which is equivalent to

$$\|Q(f, g)\|_{M^{p', q'}} \leq C \|f\|_{M^{r_1, r_2}} \|g\|_{M^{r'_1, r'_2}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

Now, one should test this estimate on families of functions f_λ, g_λ such that $Q(f_\lambda, g_\lambda)$ is concentrated inside the hyperbola $|x \cdot \omega| < 1$ (say), see Figure 1, where $\theta \asymp 1$, so that the left-hand side is comparable to $\|W(f_\lambda, g_\lambda)\|_{M^{p', q'}}$ and can be estimated from below.

The choice $f_\lambda(x) = g_\lambda(x) = e^{-\pi\lambda|x|^2}$, provides the estimate (5) when $\lambda \rightarrow +\infty$. Indeed in this case we argue exactly as in the proof of [9, Theorem 1.4]. We recall this pattern, useful also for other cases. By (14) we obtain the estimate

$$(37) \quad \|\varphi_\lambda\|_{M^{r_1, r_2}} \|\varphi_\lambda\|_{M^{r'_1, r'_2}} \lesssim \lambda^{-\frac{d}{2r'_2}} \lambda^{-\frac{d}{2r_2}}.$$

We gauge from below the norm $\|Q(\varphi_\lambda, \varphi_\lambda)\|_{M^{p', q'}}$ as follows. By taking the symplectic Fourier transform and using Lemma 2.5 and the product property (12) we have

$$\begin{aligned} \|Q(\varphi_\lambda, \varphi_\lambda)\|_{M^{p', q'}} &= \|\Theta_\sigma * W(\varphi_\lambda, \varphi_\lambda)\|_{M^{p', q'}} \\ &\asymp \|\Theta \mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{F}L^{p'}, L^{q'})} \\ &\gtrsim \|\Theta(\zeta_1, \zeta_2) \chi(\zeta_1 \zeta_2) \mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{F}L^{p'}, L^{q'})} \end{aligned}$$

for any $\chi \in C_c^\infty(\mathbb{R})$. Choosing χ supported in the interval $[-1/4, 1/4]$ and $= 1$ in the interval $[-1/8, 1/8]$, we write

$$\chi(\zeta_1 \zeta_2) = \chi(\zeta_1 \zeta_2) \Theta(\zeta_1, \zeta_2) \Theta^{-1}(\zeta_1, \zeta_2) \tilde{\chi}(\zeta_1 \zeta_2),$$

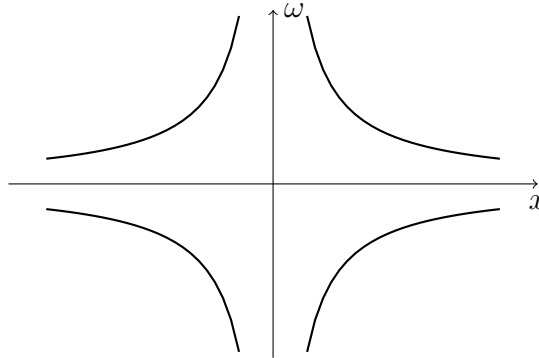


FIGURE 1. The region $|x \cdot \omega| < 1$ ($d = 1$).

with $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ supported in $[-1/2, 1/2]$ and $\tilde{\chi} = 1$ on $[-1/4, 1/4]$, therefore on the support of χ . Since by Lemma 2.5 the function $\Theta^{-1}(\zeta_1, \zeta_2)\tilde{\chi}(\zeta_1\zeta_2)$ belongs to $W(\mathcal{FL}^1, L^\infty)$, again by the product property the last expression is estimated from below as

$$\gtrsim \|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{FL}^{p'}, L^{q'})}.$$

Consider a function $\psi \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$, supported in $[-1/4, 1/4]$. Using

$$|\zeta_1\zeta_2| \leq \frac{1}{2}(|\sqrt{\lambda}\zeta_1|^2 + |\sqrt{\lambda}^{-1}\zeta_2|^2)$$

we see that $\chi(\zeta_1\zeta_2) = 1$ on the support of $\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)$, for every $\lambda > 0$.

Then, we can write

$$\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2) = \chi(\zeta_1\zeta_2)\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)$$

and by Lemma 2.2 we also infer

$$\|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\|_{W(\mathcal{FL}^1, L^\infty)} \lesssim 1$$

so that we can continue the above estimate as

$$\gtrsim \|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{FL}^{p'}, L^{q'})}.$$

Using (see e.g. [23, Formula (4.20)])

$$(38) \quad W(\varphi_\lambda, \varphi_\lambda)(x, \omega) = 2^{\frac{d}{2}}\lambda^{-\frac{d}{2}}\varphi(\sqrt{2\lambda}x)\varphi\left(\sqrt{\frac{2}{\lambda}}\omega\right),$$

we calculate

$$\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)](\zeta_1, \zeta_2) = 2^{\frac{d}{2}}\lambda^{-\frac{d}{2}}\varphi((\sqrt{2\lambda})^{-1}\zeta_2)\varphi\left(\sqrt{\frac{\lambda}{2}}\zeta_1\right),$$

so that

$$\begin{aligned} & \|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{FL}^{p'}, L^{q'})} \\ &= 2^{-d/2}\lambda^{-\frac{d}{2}}\|\psi(\sqrt{\lambda}\zeta_1)\varphi((1/\sqrt{2})\sqrt{\lambda}\zeta_1)\|_{W(\mathcal{FL}^{p'}, L^{q'})}\|\psi(\sqrt{\lambda}^{-1}\zeta_2)\varphi((\sqrt{2\lambda})^{-1}\zeta_2)\|_{W(\mathcal{FL}^{p'}, L^{q'})}. \end{aligned}$$

By Lemma 2.2 we can estimate the last expression so that

$$\|Q(\varphi_\lambda, \varphi_\lambda)\|_{M^{p', q'}} \gtrsim \lambda^{-d+\frac{d}{2p'}+\frac{d}{2q'}} \quad \text{as } \lambda \rightarrow +\infty.$$

Finally, using the estimate (37) we infer (5).

We now prove that $\max\{1/r_1, 1/r'_1\} \leq 1/q + 1/p$. If we show the estimate $1/r_1 \leq 1/q + 1/p$, then the constraint $1/r'_1 \leq 1/q + 1/p$ follows by the duality argument of Lemma 5.2. To reach this goal, we consider $f_\lambda = \varphi$ (independent

of the parameter λ) and $g = \varphi_\lambda$ as before and use the previous pattern for these families of functions, in the case $\lambda \rightarrow 0^+$. By (14) the upper estimate becomes

$$(39) \quad \|\varphi\|_{M^{r_1, r_2}} \|\varphi_\lambda\|_{M^{r'_1, r'_2}} \lesssim \lambda^{-\frac{d}{2r'_1}}.$$

The same arguments as before let us write

$$\|Q(\varphi, \varphi_\lambda)\|_{M^{p', q'}} \gtrsim \|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi, \varphi_\lambda)]\|_{W(\mathcal{F}L^{p'}, L^{q'})},$$

where $\mathcal{F}_\sigma[W(\varphi, \varphi_\lambda)]$ is computed in (22). Observe that, given any $F \in W(\mathcal{F}L^{p'}, L^{q'})$,

$$\begin{aligned} \|e^{\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2} F(\zeta_1, \zeta_2)\|_{W(\mathcal{F}L^{p'}, L^{q'})} &\gtrsim \|e^{-\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2} e^{\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2} F(\zeta_1, \zeta_2)\|_{W(\mathcal{F}L^{p'}, L^{q'})} \\ &= \|F(\zeta_1, \zeta_2)\|_{W(\mathcal{F}L^{p'}, L^{q'})}, \end{aligned}$$

because $\|e^{-\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}\|_{W(\mathcal{F}L^1, L^\infty)} \leq C$, for every $\lambda > 0$ by [9, Proposition 3.2]. So, writing

$$c_\lambda = \frac{1}{(\lambda+1)^{\frac{d}{2}}}$$

(notice $c_\lambda \rightarrow 1$ for $\lambda \rightarrow 0^+$) we are reduced to

$$\|Q(\varphi, \varphi_\lambda)\|_{M^{p', q'}} \gtrsim c_\lambda \|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2}\|_{W(\mathcal{F}L^{p'}, L^{q'})} \|\psi(\sqrt{\lambda}^{-1}\zeta_2)e^{-\frac{\pi}{\lambda+1}\zeta_2^2}\|_{W(\mathcal{F}L^{p'}, L^{q'})}.$$

By Lemma 2.2 we can estimate, for $\lambda \rightarrow 0^+$,

$$\|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2}\|_{W(\mathcal{F}L^{p'}, L^{q'})} = \|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi}{\lambda+1}(\sqrt{\lambda}\zeta_1)^2}\|_{W(\mathcal{F}L^{p'}, L^{q'})} \asymp \lambda^{-\frac{d}{2q'}},$$

whereas

$$\begin{aligned} \|\psi(\sqrt{\lambda}^{-1}\zeta_2)e^{-\frac{\pi}{\lambda+1}\zeta_2^2}\|_{W(\mathcal{F}L^{p'}, L^{q'})} &= \lambda^{\frac{d}{2}}(\lambda+1)^{\frac{d}{2}} \left\| \int \hat{\psi}(\sqrt{\lambda}(\zeta_2 - \eta)) e^{-\pi(\lambda+1)|\eta|^2} d\eta \right\|_{L^{p'}} \\ &= \lambda^{\frac{d}{2}}(\lambda+1)^{\frac{d}{2}} \lambda^{-\frac{d}{2p'}} \left\| \int \hat{\psi}(x - \sqrt{\lambda}\eta) e^{-\pi(\lambda+1)|\eta|^2} d\eta \right\|_{L^{p'}} \\ &= (\lambda+1)^{\frac{d}{2}} \lambda^{-\frac{d}{2p'}} \left\| \int \hat{\psi}(x - t) e^{-\pi \frac{\lambda+1}{\lambda}|t|^2} dt \right\|_{L^{p'}} \\ &= \lambda^{\frac{d}{2} - \frac{d}{2p'}} \|\hat{\psi} * K_{1/\sqrt{\lambda}}\|_{L^{p'}} \\ &\sim \lambda^{\frac{d}{2} - \frac{d}{2p'}} \|\hat{\psi}\|_{p'}, \text{ as } \lambda \rightarrow 0^+ \end{aligned}$$

where $K_{1/\sqrt{\lambda}}(\zeta_2) = \lambda^{-\frac{d}{2}}(\lambda+1)^{\frac{d}{2}} e^{-\frac{\pi(\lambda+1)}{\lambda}|\zeta_2|^2}$, $\lambda \rightarrow 0^+$, is an approximate identity. So that

$$\lambda^{-\frac{d}{2r'_1}} \gtrsim \lambda^{-\frac{d}{2q'}} \lambda^{\frac{d}{2p'}}$$

and, for $\lambda \rightarrow 0^+$, we obtain

$$\frac{1}{r_1} \leq \frac{1}{q} + \frac{1}{p},$$

as desired.

It remains to prove that $\max\{1/r_2, 1/r'_2\} \leq 1/q + 1/p$. Again, it is enough to show that $1/r_2 \leq 1/q + 1/p$ and invoke Lemma 5.2 for $1/r'_2 \leq 1/q + 1/p$.

An explicit computation (see [12, Proposition 5.3]) shows that

$$(40) \quad \mathcal{F}^{-1} \text{Op}_W(\sigma) \mathcal{F} = \text{Op}_W(\sigma \circ J),$$

where $J(x, \omega) = (\omega, -x)$ as defined in (20) (this is also a consequence of the intertwining property of the metaplectic operator \mathcal{F} with the Weyl operator $\text{Op}_W(\sigma)$ [15, Corollary 221]).

Now, observing that $\Theta_\sigma \circ J = \Theta_\sigma$, we obtain

$$\begin{aligned} (a * \Theta_\sigma)(Jz) &= \int_{\mathbb{R}^{2d}} a(u) \Theta_\sigma(Jz - u) du = \int_{\mathbb{R}^{2d}} a(u) \Theta_\sigma(J(z - J^{-1}u)) du \\ &= \int_{\mathbb{R}^{2d}} a(u) \Theta_\sigma(z - J^{-1}u) du = \int_{\mathbb{R}^{2d}} a(Ju) \Theta_\sigma(z - u) du \\ &= (a \circ J) * \Theta_\sigma(z). \end{aligned}$$

The previous computations together with (40) gives

$$\mathcal{F}^{-1} \text{Op}_{\text{BJ}}(a) \mathcal{F} = \mathcal{F}^{-1} \text{Op}_{\text{BJ}}(a \circ J) \mathcal{F}.$$

On the other hand, the map $a \mapsto a \circ J$ is an isomorphism of $M^{p,q}$, so that (36) is in fact equivalent to

$$(41) \quad \|\text{Op}_{\text{BJ}}(a)f\|_{W(\mathcal{F}L^{r_1}, L^{r_2})} \lesssim \|a\|_{M^{p,q}} \|f\|_{W(\mathcal{F}L^{r_1}, L^{r_2})} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), f \in \mathcal{S}(\mathbb{R}^d).$$

The estimate (41) can be written as

$$|\langle a, Q(f, g) \rangle| \leq C \|a\|_{M^{p,q}} \|f\|_{W(\mathcal{F}L^{r_1}, L^{r_2})} \|g\|_{W(\mathcal{F}L^{r'_1}, L^{r'_2})} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), f, g \in \mathcal{S}(\mathbb{R}^d)$$

which is equivalent to

$$\|Q(f, g)\|_{M^{p',q'}} \leq C \|f\|_{W(\mathcal{F}L^{r_1}, L^{r_2})} \|g\|_{W(\mathcal{F}L^{r'_1}, L^{r'_2})} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

Now, taking $f = \varphi$ and $g = \varphi_\lambda$ as before, we observe that, for $\lambda \rightarrow 0^+$, by (14),

$$\|\varphi_\lambda\|_{W(\mathcal{F}L^{r'_1}, L^{r'_2})} \asymp \lambda^{-\frac{d}{2}} \|\varphi_{1/\lambda}\|_{M^{r'_1, r'_2}} \asymp \lambda^{-\frac{d}{2} + \frac{d}{2r_2}} = \lambda^{-\frac{d}{2r_2}}.$$

Arguing as in the previous case we obtain $1/r_2 \leq 1/q + 1/p$. This concludes the proof. \square

ACKNOWLEDGEMENTS

The first author was partially supported by the Italian Local Project “Analisi di Fourier per equazioni alle derivate parziali ed operatori pseudo-differenziali”, funded by the University of Torino, 2013. The second author was supported by the FWF grant P 2773.

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